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Head-on collision between two solitary waves in a Rayleigh-Bénard convecting fluid

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A head-on collision between two solitary waves from opposite directions in a Rayleigh-Bénard convecting fluid is investigated by use of the Poincaré-Lighthill-Kuo method. The fluid system is bounded below by an isothermic plane and above a free deformable surface on which a heat flux is fixed. The results show that, near the transition of the long-wavelength oscillatory instability, the solitary waves emerging from the collision can preserve their original identities to the second order. The phase shifts due to the collision are calculated explicitly.

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It is well known that the long-time asymptotic behavior of two-dimensional unidirectional shallow water waves in the case of weak nonlinearity is described by the Korteweg–de Vries (KdV) equation [1]. Since the inverse scattering transform (IST) for exactly solving the KdV equation was found by Gardner, Greene, Kruskal, and Miura [2], the interesting features of the collision between solitary waves have been revealed: When two solitary waves approach closely, they interact, exchange their energies and positions with one another, and then separate off, regaining their original wave forms. Throughout the whole process of the collision, the solitary waves are remarkably stable entities, preserving their identities through interaction. The unique effect due to the collision is their phase shifts [3]. It is believed that this striking colliding property of solitary waves can only be preserved in a conservative system.

In recent years, much attention has been paid to the study of the convecting fluid whose first instability from the static state is oscillatory. Examples include the binary fluid convection [4], the electrohydrodynamic convection in nematic liquid crystals [5], etc. A common feature of these systems is that the transition occurs with finite wave number and frequency. General arguments [6], as well as asymptotic expansion in particular cases [7], show that the nonlinear behavior near the transition is governed by coupled Landau-Newell-type equations. Traveling waves are governed by the Ginzberg-Landau equations. Much work has been devoted to the study of the solution to these equations and their comparison with experimental results [8,9]. Recently, Benguria and Depassier [10] investigated the oscillatory instability in the Rayleigh-Bénard convecting fluid with a free surface. A new type of oscillatory instability whose transition occurs at vanishing wave number and frequency, i.e., a long-wavelength oscillatory instability, was found. The nonlinear evolution of the surface wave at the transition was shown to satisfy the KdV equation [11–13], which is completely integrable. This is a quite remarkable result because the Rayleigh-Bénard convecting fluid under study is a thermally driven dissipative system. Naturally one may ask: How about the collision between the solitary waves in this nonconservative system? This paper will give an answer.

According to the IST, all KdV solitary waves travel in the same direction (under the boundary condition vanishing at infinity) [2,3], so for the overtaking collision between solitary waves in the Rayleigh-Bénard convecting fluid, one can use the IST to obtain the overtaking colliding effect of the solitary waves. However, for the head-on collision between two solitary waves, we must employ some kind of asymptotic expansion to solve the original Navier-Stokes equations [14]. Based on the work of Su and Mirie [15], in this paper we shall use the Poincaré-Lighthill-Kuo (PLK) method to investigate the head-on collision between two solitary waves traveling in opposite directions in the Rayleigh-Bénard convecting fluid.

Let us consider a layer of fluid that, at rest, lies between $z=0$ and d. Upon it acts a gravitational field $\mathbf{g}=-g\hat{\mathbf{z}}$. The fluid is described by the Boussinesq equations [11],

$$\nabla\cdot\mathbf{v}=0\ , \tag{1}$$

$$\rho_0\frac{d\mathbf{v}}{dt}=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{g}\rho\ , \tag{2}$$

$$\frac{dT}{dt}=\kappa\nabla^2 T\ , \tag{3}$$

$$\rho=\rho_0[1-\alpha(T-T_0)]\ , \tag{4}$$

where $d/dt=\partial/\partial t+\mathbf{v}\cdot\nabla$ is the convection derivative; p, T, ρ, and $\mathbf{v}$ denote the pressure, temperature, density, and fluid velocity, respectively. The fluid's properties, that is, its viscosity μ, thermal diffusivity κ, and coefficient of thermal expansion α, are constants. Furthermore, we restrict ourselves to two-dimensional motion, so that $\mathbf{v}=(u,0,w)$.

The fluid is bounded above by a free surface in contact with a passive gas, which exerts upon it a constant pressure p_a, and below a plane surface. As motion sets in, the free surface is deformed; we shall write its position as $z=d+\eta(x,t)$. The boundary conditions on the upper surface are [11]

$$\eta_t+u\eta_x=w\ , \tag{5}$$

$$p-p_a-(2\mu/N^2)[w_z+u_x\eta_x^2-\eta_x(u_z+w_x)]=0\ , \tag{6}$$

$$\mu(1-\eta_x^2)(u_z+w_x)+2\mu\eta_x(w_z-u_x)=0\ , \tag{7}$$

$$\hat{\mathbf{n}}\cdot\nabla T=-F/k\ , \tag{8}$$

on $z=d+\eta$. Here $N=(1+\eta_x^2)^{1/2}$, $\hat{\mathbf{n}}=(-\eta_x,0,1)/N$ is the unit normal to the free surface, F is the prescribed normal heat flux, and k is the thermal conductivity. Denoting by T_b the fixed temperature of the lower surface, the boundary conditions on the lower surface $z=0$ are

$$w=u_z=0,\quad T=T_b\ . \tag{9}$$

The static solution to these equations is given by $T_s=-F(z-d)/k+T_0$, $\rho_s=\rho_0[1+(\alpha F/k)(z-d)]$, and $p=p_a-g\rho_0[(z-d)+(\alpha F/2k)(z-d)^2]$. We have chosen the reference temperature T_0 as the value of the static temperature on the upper surface. The temperature on the lower surface is then $T_b=T_0+Fd/k$. We shall adopt d as the unit of length, d^2/κ as the unit of time, $\rho_0 d^3$ as the unit of mass, and Fd/k as the unit of temperature. Then there are three dimensionless parameters involved in the problem, the Prandtl number $\sigma=\mu/(\rho_0\kappa)$, the Rayleigh number $R=\rho_0 g\alpha F d^4/(\kappa k\mu)$, and the Galileo number $G=gd^3\rho_0^2/\mu^2$.

In order to study the head-on collision between two solitary waves from opposite directions in this thermally driven dissipative system, we use the PLK method [15]. We anticipate that the collision will result in post-interaction phase changes of them. So we introduce the following transformation of wave-framed coordinates with phase functions:

$$r=\epsilon(x-ct)+\epsilon^2\phi_0(l,\tau)+\epsilon^3\phi_1(r,l,\tau)+\cdots\ , \tag{10}$$

$$l=\epsilon(x+ct)+\epsilon^2\psi_0(r,\tau)+\epsilon^3\psi_1(r,l,\tau)+\cdots\ , \tag{11}$$

$$\tau=\epsilon^3 t\ , \tag{12}$$

where ϕ_j and ψ_j ($j=0,1,2,\ldots$) are to be determined in the process of our perturbation solution of (1)–(9). We introduce the asymptotic expansions $u=\epsilon^2(u_0+\epsilon u_1+\epsilon^2 u_2+\cdots)$, $w=\epsilon^3(w_0+\epsilon w_1+\epsilon^2 w_2+\cdots)$, $p-p_s=\epsilon^2(p_0+\epsilon p_1+\epsilon^2 p_2+\cdots)$, $T-T_s=\epsilon^3(\theta_0+\epsilon\theta_1+\epsilon^2\theta_2+\cdots)$, and $\eta=\epsilon^2(\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\cdots)$. Then we can proceed to solve each order in ϵ.

To leading order [$O(1)$ order] the asymptotic solutions to (1)–(9) yield

$$u_0=f(r,\tau)-q(l,\tau),\quad w_0=-(f_r-q_l)z\ ,$$

$$p_0=\frac{G\sigma^2}{c}(f+q),\quad \theta_0=(f_r-q_l)T_0(z),\quad \eta_0=\frac{1}{c}(f+q)\ ,$$

where $f=f(r,\tau)$ and $q=q(l,\tau)$ are two functions to be determined in the next orders. The definition of the functions $T_j(z)$ and $P_j(z)$ ($j=0,1,2,3,\ldots$) appearing in this and the next orders (see below) have been given in the Appendix. We must point out that the vanishing boundary condition at infinity has been used (i.e., the fluid is static when $r^2+l^2\to\infty$) in obtaining the above solution, the same as in Refs. [11–13].

At order $O(\epsilon)$, we find

$$u_1=g(r,l,\tau),\quad w_1=-(g_r+g_l)z\ ,$$

$$\eta_1=-\frac{1}{c}(\partial_r-\partial_l)^{-1}(g_r+g_l)\ ,$$

$$p_1=-2\sigma(f_r-q_l)+\frac{G\sigma^2}{c}(\partial_r-\partial_l)^{-1}(g_r+g_l)+\sigma R(f_r-q_l)P_1(z)\ ,$$

$$\theta_1=-c(f_{rr}+q_{ll})T_1(z)+(g_r+g_l)T_0(z)\ ,$$

where $g=g(r,l,\tau)$ is an undetermined function and $(\partial_r-\partial_l)^{-1}$ is the inverse operator of $(\partial_r-\partial_l)$. The solubility condition $u_{1z}(1)=\int_0^1 u_{1zz}dz$ determines the critical wave speed

$$c^2=G\sigma^2\ . \tag{13}$$

In the case of $O(\epsilon^2)$ order, the solubility condition $u_{2z}(t)=\int_0^1 u_{2zz}dz$ determines the critical Rayleigh number $R=R_c=30$, i.e., the same as in Ref. [11]; and $g=f_1(r,\tau)-q_1(l,\tau)$. The solution is

$$u_2=h(r,l,\tau)-\frac{c}{\sigma}(f_{1r}+q_{1l})P_0(z)+\frac{G\sigma}{c}(\partial_r-\partial_l)^{-1}(f_{1rr}-q_{1ll})P_0(z)+(f_{rr}-q_{ll})[-3P_0(z)+R_cP_2(z)]\ ,$$

$$p_2=G\sigma^2\eta_2+\frac{1}{2c^2}\sigma R_c(f+q)^2-2\sigma(f_{1r}-q_{1l})-\frac{\sigma}{c}Z(f_{rr}+q_{ll})-c(f_{rr}+q_{ll})\left[\frac{G\sigma^2}{c^2}P_0(z)+\sigma R_cP_2(z)\right]+\sigma R_c(f_{1r}-q_{1l})P_1(z)\ ,$$

$$\eta_2=\frac{1}{c}(\partial_r-\partial_l)^{-1}\left[2(\phi_{0l}f_r-\psi_{0r}q_l)+\frac{2}{c}(ff_r-qq_l)+\frac{1}{c}(f_\tau+q_\tau)+(\partial_r-\partial_l)h+\frac{c}{3\sigma}(f_{1rr}+q_{1ll})-\frac{G\sigma}{3c}(\partial_r-\partial_l)^{-1}(f_{1rrr}-q_{1lll})+(f_{rrr}-q_{lll})\left[1-\frac{17R_c}{315}\right]\right]\ ,$$

where h is an undetermined function and the expressions for w_2 and θ_2 are omitted here.

At $O(\epsilon^3)$, the solubility condition $u_{3z}(1)=\int_0^1 u_{3zz}dz$ yields the equations for ϕ_0, ψ_0, f, and q as the following:

$$\phi_{0l}(l,\tau)=\chi q(l,\tau)\ , \tag{14}$$

$$\psi_{0l}(r,\tau)=\chi f(r,\tau)\ , \tag{15}$$

$$f_\tau+\lambda_1 f f_r+\lambda_2 f_{rrr}=0\ , \tag{16}$$

$$-q_\tau+\lambda_1 q q_l+\lambda_2 q_{lll}=0\ , \tag{17}$$

where

$$\chi=\frac{1}{4c}\left[1-\frac{30}{G\sigma}\right]\ , \tag{18}$$

$$\lambda_1=\frac{3(10+G\sigma)}{2G\sigma}\ , \tag{19}$$

$$\lambda_2=\tfrac{1}{2}G\sigma^2[\tfrac{1}{3}+\tfrac{34}{21}\sigma]\ . \tag{20}$$

From (14) and (15) we have the solution of the phase function at $O(\epsilon^2)$:

$$\phi_0=\int_{-\infty}^{l} q(l',\tau)dl',\quad \psi_0=\int_{\infty}^{r} f(r',\tau)dr'\ . \tag{21}$$

The evolution equations for $f(r,\tau)$ and $q(l,\tau)$ satisfy the KdV equation, (16) and (17), respectively. Higher-order corrections, which do not interest us here, determine evolution equations for the functions g and h. The single soliton solutions of (16) and (17) are

$$f=A\,\mathrm{sech}^2\left[\left[\frac{\lambda_1 A}{12\lambda_2}\right]^{1/2}(r-\tfrac{1}{3}\lambda_1 A\tau-\phi_A)\right]$$

$$(\text{right-running soliton})\ , \tag{22}$$

$$q=B\,\mathrm{sech}^2\left[\left[\frac{\lambda_1 B}{12\lambda_2}\right]^{1/2}(l+\tfrac{1}{3}\lambda_1 B\tau-\phi_B)\right]$$

$$(\text{left-running soliton})\ , \tag{23}$$

where A, ϕ_A, B, and ϕ_B are integral constants. From this we can calculate the phase changes due to their collision as

$$\phi_0=\chi\left[\frac{12\lambda_2 B}{\lambda_1}\right]^{1/2}\left\{\tanh\left[\left[\frac{\lambda_1 B}{12\lambda_2}\right]^{1/2}(l+\tfrac{1}{3}\lambda_1 B\tau-\phi_B)\right]+1\right\}\ , \tag{24}$$

$$\psi_0=\chi\left[\frac{12\lambda_2 A}{\lambda_1}\right]^{1/2}\left\{\tanh\left[\left[\frac{\lambda_1 A}{12\lambda_2}\right]^{1/2}(r-\tfrac{1}{3}\lambda_1 A\tau-\phi_A)\right]-1\right\}\ . \tag{25}$$

We finally obtain the expression of the surface wave up to $O(\epsilon^2)$ as follows:

$$\eta=\epsilon^2\left\{A\,\mathrm{sech}^2\left[\left[\frac{\lambda_1 A}{12\lambda_2}\right]^{1/2}(r-\tfrac{1}{3}\lambda_1 A\tau-\phi_A)\right]+B\,\mathrm{sech}^2\left[\left[\frac{\lambda_1 B}{12\lambda_2}\right]^{1/2}(l+\tfrac{1}{3}\lambda_1 B\tau-\phi_B)\right]\right\}+O(\epsilon^3)\ , \tag{26}$$

$$r=\epsilon(x-ct)+\epsilon^2\chi\left[\frac{12\lambda_2 B}{\lambda_1}\right]^{1/2}\left\{\tanh\left[\left[\frac{\lambda_1 B}{12\lambda_2}\right]^{1/2}(1+\tfrac{1}{3}\lambda_1 B\tau-\phi_B)\right]+1\right\}\ , \tag{27}$$

$$l=\epsilon(x-ct)+\epsilon^2\chi\left[\frac{12\lambda_2 A}{\lambda_1}\right]^{1/2}\left\{\tanh\left[\left[\frac{\lambda_1 A}{12\lambda_2}\right]^{1/2}(r-\tfrac{1}{3}\lambda_1 A\tau-\phi_A)\right]-1\right\}\ . \tag{28}$$

The existence of undamped solitary waves in this thermally driven dissipative system is possibly due to the fact that the energy released by buoyancy balances exactly the amount of kinetic energy dissipated by viscosity. This can be shown as in Ref. [11].

Now from (26)–(28) we can estimate the phase shifts in the collision process of two solitons traveling in opposite directions. Let us assume that the soliton f (denoted by A) and the soliton q (denoted by B) are at a long distance from each other at an initial instant ($t=-\infty$); i.e., soliton A is at $l=-\infty$ and $r=0$, and B is at $r=\infty$ and $l=0$. After collision ($t=\infty$), A is far to the right of B; i.e., A is at $l=\infty$ and $r=0$, and B is at $r=-\infty$ and $l=0$. In this case, the phase shifts of A and B, δ_A and δ_B, are given by

$$\delta_A=\epsilon(x-ct)|_{r=0,\,l=\infty}-\epsilon(x-ct)|_{r=0,\,l=-\infty}$$
$$=2\epsilon^2\chi\left[\frac{12\lambda_2 B}{\lambda_1}\right]^{1/2}\ , \tag{29}$$

$$\delta_B=\epsilon(x+ct)|_{r=-\infty,\,l=0}-\epsilon(x+ct)|_{r=\infty,\,l=0}$$
$$=2\epsilon^2\chi\left[\frac{12\lambda_2 A}{\lambda_1}\right]^{1/2}\ . \tag{30}$$

In principle, higher-order effects due to collision can be calculated further by using the above formulation.

In conclusion, we have investigated the head-on collision between left-running and right-running solitary waves in the Rayleigh-Bénard convecting fluid by using the PLK method. The results show that the solitary waves emerging from the collision preserve their original identities to $O(\epsilon^2)$. The phase shifts due to the collision have been calculated explicitly. It is remarkable that the evolution of weak nonlinear surface waves near the transition in this *thermally driven dissipative system* is similar to that of a conservative one.

We make note of the recent work by Weidman, Linde, and Velarde [16], which gives considerable evidence of soliton behavior in both heat-transfer- and mass-transfer-driven Marangoni-Bénard convection flows. A constant phase shift is observed as a result of the head-on collision of two solitary waves traveling in opposite directions. A theoretical explanation for the results of this experiment is needed. The study is still underway.

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APPENDIX: THE FUNCTIONS $T_j(z)$ AND $P_j(z)$

The functions $T_j(z)$ and $P_j(z)$ are defined by

$$T_j(z)=\int_0^z dz_1\int_1^{z_1} dz_2 T_{j-1}(z_2)=\int_0^z dz_1 P_j(z_1) , \quad \text{(A1)}$$

$$P_j(z)=\int_1^z dz_1 T_{j-1}(z_1) , \quad \text{(A2)}$$

$$T_{-1}(z)=z , \quad \text{(A3)}$$

$j=0,1,2,\ldots$. From (A1)–(A3) we have

$$T_0(z)=\frac{1}{3!}(z^3-3z) , \quad \text{(A4)}$$

$$T_1(z)=\frac{1}{5!}(z^5-10z^3+25z) , \quad \text{(A5)}$$

$$T_2(z)=\frac{1}{7!}(z^7-21z^5+175z^3-427z) , \quad \text{(A6)}$$

$$\vdots$$

$$P_0(z)=\frac{1}{2!}(z^2-1) , \quad \text{(A7)}$$

$$P_1(z)=\frac{1}{4!}(z^4-6z^2+5) , \quad \text{(A8)}$$

$$P_2(z)=\frac{1}{6!}(z^6-15z^4+75z^2-61) , \quad \text{(A9)}$$

$$\vdots$$

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